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Momenta and reduction for general relativity

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Abstract. *A slice theorem for the action of* Diff *on the space of solutions of the Einstein equations in the asymptotically flat case is proved.*

1. INTRODUCTION

This is the first of two papers aimed at analyzing the ADM momentum for General Relativity in terms of symplectic geometry, momentum mappings and reduction, and carrying the work of Regge and Teitelboim [37] to its logical conclusion. This part discusses the proof of the slice theorem for the action of the group of diffeomorphisms asymptotic to Poincaré transformations on the set of asymptotically flat solutions (in the sense of spatial infinity) of the Einstein equations. The proof is in the context of spatial infinity and maximal slicings. Eventually, one hopes to extend this type of analysis also to null infinity and constant mean curvature slicings.

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1.1. Reduction of dynamical systems

The conceptual background of this work is as follows. Let (P, ω) be a symplectic manifold and let the Lie group G act on P. Assume that the action of G on M admits a momentum mapping, i.e. a map $J : P \rightarrow \mathbf{g}^*$, where $\mathbf{g}^*$ is the dual of the Lie algebra of G, such that $< J, \xi >$ is the Hamilotnian function generating the action on M corresponding to $\xi \in \mathbf{g}$. The following is known (under certain technical conditions):

Level Sets

The space $J^{-1}(\xi)$ is a manifold except at points where the action of G has a nontrivial isotropy group, where at most quadratic singularities occur.

Reduction

The space $J^{-1}(\xi)/G_\xi$ is a stratified symplectic manifold with at most quadratic singularities, the space of dynamical degrees of freedom. Here, G_ξ denotes the isotropy group of ξ w.r.t. the coadjoint action of G on $\mathbf{g}^*$.

For information about the general situation, see [2].

1.2. The Spatially Compact Case

The above programme has been carried out for general relativity in the spatially compact case. Let $V = M \times \mathbf{R}$ where M is some compact, orientable 3-dimensional manifold and let $\mathbf{M}$ denote the space of Riemannian metrics on M. Let g be a Lorentz-metric on V satisfying $Ein(g) = 0$. The group in this case is $Diff(V)$, the group of diffeomorphisms of V and the momentum map is the ADM super-momentum and super-Hamiltonian:

$$\Phi = (H, J) : T^*\mathbf{M} \rightarrow \Lambda^0_d \times \Lambda^1_d = \text{«}\textit{dual of space of Lapses} \text{ and } \textit{Shifts}\text{»}.$$

Here, Λ^0_d and Λ^1_d denote the spaces of function densities and 1-form densities on M, respectively. The space of solutions to Einsteins equations is fibered over $\Phi^{-1}(0)$ which is smooth at (γ, π) if and only if the initial data (γ, π) corresponds to a solution g which has no Killing-field. The reduced space $\Phi^{-1}(0)/Diff_V$ of «true dynamical degrees of freedom» has been constructed [28] and turns out to be a stratified symplectic ILH manifold.

It should be noted that in the case of general relativity, the action of $Diff(V)$ on the constraint set $\mathbf{C}$ is not a true group action. In fact, the Poisson brackets of the constraints are of the form

$$\{\Phi_\mu(\gamma, \pi)(x), \Phi_\upsilon(\gamma, \pi)(x')\} = C^\alpha_{\mu\upsilon}(\gamma, \pi)\delta_\alpha(x, x'),$$

using physics notation. See [39, p. 545] for the exact expression. The important fact to note is that the «structure functions» $C^{\alpha}_{\mu\nu}(\gamma, \pi)$ are indeed functions and not constants as is the case in ordinary gauge theories such as Yang-Mills.

It is therefore necessary to check some of the statements about symplectic structure on the reduced space etc. «by hand», see [24] for a discussion in the spatially compact case.

1.3. The asymptotically flat case

Here, the appropriate group is the group of those diffeomorphisms which preserve the conditions for asymptotic flatness. The nature of the gauge group in the spatial infinity case depends strongly on the precise asymptotic conditions used.

Apart from the compactification schemes of Ashtekar-Hansen and Geroch among others, three main types of asymptotic conditions have been studied: the finite energy condition [35], the York Quasi Isotropic gauge conditions [43] and conditions of the type introduced by Regge and Teitelboim [37].

These turn out to have quite different properties. For the finite energy condition, one finds that the group which leaves the asymptotic conditions invariant is a semidirect product $S \ltimes L$ where S consists of diffeomorphisms η such that roughly $D^2\eta \in L^2$, which means that S contains translations and time translations (i.e. $(N, X) \to$ constant). Under these conditions, it does not appear to be possible to talk about Hamiltonian dynamics. For a general element of $Lie(S \ltimes L)$, the corresponding momentum integral does not converge, although for the special case of translations and time translations, the ADM-momentum is well defined.

The York QI gauge condition as formulated in [43] has the desirable feature that no «supertranslations» are allowed, but a more detailed analysis reveals that without extra conditions, the transformations corresponding to boosts are not well-behaved. In any case, the QI asymptotic conditions do not give a well-defined angular and boost momentum and therefore are suitable only for the study of diffeomorphisms asymptotic to translations and time translations.

To allow a study of the question of momentum w.r.t. rotations and boosts, we introduce in §3 a set of conditions which are a combination of those introduced in [37] and the QI conditions of York. Conditions of the type introduced in [37] were recently studied by Beig and O-Murchadha [10] and were by them termed «parity conditions» which is the name that will be used here.

The space of transformations $Diff_P$ which leaves invariant the space of solutions to the Einstein equations satisfying the parity conditions is a semidirect product $Diff_P = Diff_S \ltimes P$, where P denotes the Poincaré group and $Diff_S$ denotes the space of diffeomorphisms which are asymptotic to supertranslations, which in this case are $O(1)$ with odd leading term.

When the QI conditions are added, the $Diff_S$ part is restricted to $Diff_I$, the space of diffeomorphisms which tend to the identity at $x = \infty$. For details, see §2. In the following discussion, we will often assume that the QI conditions hold, although many of the statements are correct without this condition.

1.4. The Momentum Mapping for General Relativity

In the spatially compact case, Φ is the momentum-mapping for the action of *Diff* on the phase space of General Relativity. As pointed out by Regge and Teitelboim [37], this is no longer true in the noncompact case for Lapse and Shift not tending to 0 at infinity. It can be shown that when using the parity conditions, the Lapses and Shifts corresponding to the group of supertranslations S have zero momenta. Thus, assuming the QI conditions, we see the ADM momentum appearing as the momentum-mapping w.r.t. the 2:nd component of the semidirect product $Diff_I \ltimes P$.

Note that from this point of view, the classical form of the ADM-momentum is correct only using restrictive assumptions, such as those in [37], which are nontrivial restrictions not only on the gauge freedom but also on the degrees of dynamical freedom of the gravitational field. Call the total momentum mapping $\Phi_E = \Phi + E$. Then $\Phi_E : T^*\mathbf{M} \to (\Lambda_d^0 \times \Lambda_d^1) \times \mathbf{p}^*$ where $\Lambda_d^0 \times \Lambda_d^1$ is the $3 + 1$ version of the dual of the Lie algebra of $Diff_I$ and $\mathbf{p}^*$ is the dual of the Lie algebra $\mathbf{p}$ of the Poincaré group. E will consist of certain integrals over spheres at infinity. From the general theory [2], one expects the whole programme scetched in §1.1 (level-sets, reduction) to apply to Φ_E. For example, one expects that (assuming the QI conditions)

$$\Phi^{-1}(0)/Diff_I$$

is a symplectic *manifold* (no isometries can be in $Diff_I$) and that for $\xi \in \mathbf{p}^*$, the spaces $\Phi_E^{-1}(0 \times \xi)$ and

$$\Phi_E^{-1}(0 \times \xi)/Diff_{P_\xi},$$

are manifolds except at points corresponding to flat space or spaces with rotational symmetries. Here, $Diff_{P_\xi} = Diff_I \ltimes P_\xi$, where P_ξ denotes the isotropy group of ξ in P. Note here that $\Phi_E^{-1}(0 \times \xi)$ is a smooth variety in a H^s setting, but to construct a C^∞ structure on $\Phi_E^{-1}(0 \times \xi)/Diff_{P_\xi}$, we have to use ILH structures.

1.5. Discussion of Reduction of Einsteins Equations

Reduction of a system with gauge degrees of freedom may be performed either by explicitly introducing a «gauge-fixing», i.e. introducing a slice and changing the Hamiltonian so that the dynamics stays in the slice or, by what may be termed «intrinsic reduction» where one passes to the group theoretical quotient of the (constrained) phase space by the action of the gauge group. This method, which was introduced by J.E. Marsden and A. Weinstein [31] is applicable under quite general conditions, but does not give an explicit set of «reduced variables», which are useful in calculations.

No satisfactory solution to the problem of constructing reduced variables for G.R. has been found, except in the spatially compact case of systems with one spatial Killing field in the work of V. Moncrieff [32]. This is one of the important open problems in G.R.

In this paper we will be concerned with the intrinsic reduction of Einsteins equations in the asymptotically flat case, which does not appear to have been carried out in detail before, although the problem has been discussed in various contexts [28], [16, § 10]. To give a background to the problem we will here discuss the approaches to the question of reduction of Einsteins equations which have been made earlier.

P.A.M. Dirac introduced a general method for computing the «reduced Hamiltonian» and applied it to General Relativity in [22]. Dirac noted that the Hamiltonian constraint may be effectively reduced by separating the variables into a conformal metric $\tilde{\gamma}(\det \tilde{\gamma} \equiv 1)$, a conformal factor ϕ, a conformally rescaled trace free part of the momentum $\tilde{\pi}$ and tr$K$, the trace of the 2:nd fundamental form of the 3-surface M. Here, tr$K$ plays the role of time variable and ϕ plays the role of Hamiltonian, dual to tr$K$. This idea has been extensively discussed and generalized in a series of papers by J. York (see [34] and references therein).

Shortly after the Einstein Equations were put on Hamiltonian form by Dirac, the reduction problem was taken up by Arnowitt, Deser and Misner (ADM) who in a series of papers (see [3] and references therein) discussed the Hamiltonian form and reduction of the Einstein equations. The Hamiltonian form introduced by Dirac was refined and the Hamiltonian version of the Hilbert Lagrangian was derived.

The general method applied by ADM was first introduced by J. Schwinger in a Quantum Field Theory context and is known as the Schwinger variational principle. In the work of ADM, a set of variables (γ_{TT}, π_{TT}) were introduced which serve as dynamical degrees of freedom. These are defined in terms of a decomposition of (γ, π) w.r.t. a fixed flat background metric, which makes the procedure non-covariant. The decomposition introduced by ADM has been further studied in the work of Deser [21] and York [42]. Although a great deal of insight into the structure of the constraint equations has been derived in the course of these investigations, no general solution to the reduction problem has been achieved.

Finally, the possibility of performing a manifestly «covariant» reduction (without 3 + 1-ing the equations) is indicated by the work on covariant brackets by Marsden et. al. [30]. One might also mention the approach to the Yang-Mills equations taken by Gross [26], which if it generalizes to the case of General

Relativity (as indicated by the work of Mandelstam [29]) may give some new insight into the problem.

1.6. Statement of the Slice Theorem

Let G denote $Diff_P^{s+1}$ or $Diff_I^{s+1}$, let $A : G \times Ein_{QI}^{s+l}(V) \to Ein_{QI}^{s+l}(V)$ denote the action (by pullback) on the space of solutions of the Einstein equations satisfying the parity and QI conditions and for $l \geqslant 1$, let $O_G^s(g) = A(G, g)$ denote the orbit of $g \in Ein_{QI}^{s+l}(V)$ under the action of $Diff_P$ or $Diff_I$.

The statement of the slice theorem for the action of G on $Ein_{QI}(V)^s$ is as follows (cf. [28, Theorem 4.1] for the spatially compact case).

THEOREM 1.1. *The following holds for* $g_0 \in Ein_{QI}^{s+l}(V)$.

a) The orbit $O_G^s(g_0)$ *is a closed* C^l *embedded submanifold of the manifold* $Ein_{QI}^s(V)$.

b) There exists a submanifold $S_{g_0}^s$ *containing* g_0 *which is a slice for the action (for the details of the definition of a slice, see [1, Definition 4.1], [28, Theorem 4.1]) or Appendix B.* ∎

Remark. We will prove the theorem assuming the validity of the conjectures on global existence of solutions and maximal slicings. It is clearly possible to formulate «local» (in time) slice theorems similar to those in [28], which are valid under weaker assumptions on the existence of maximal slicings, but we will not pursue this here. ∎

The action of the group of asymptotically Euclidean diffeomorphisms on the space of asymptotically Euclidean metrics on $\mathbf{R}^n$ was analyzed by the author [1] and plays an important role in the proof of Theorem 1.

The notation which will be used in the following is fairly standard and follows [28]. To make the presentation in the main part of the paper more coincise, some technical details have been deferred to two appendices. In Appendix A, some details of analysis on spaces of radially smooth functions is given and in Appendix B, some background to the slice theorem for the asymptotically euclidean case with radial smoothness is given.

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2. THE PARITY CONDITIONS

In order to formulate the asymptotic conditions which will be used in this paper, we introduce some spaces of functions with radial smoothness. Let H^s_δ denote the weighted L^2 Sobolev space with norm $\| \cdot \|_{s,\delta}$ (see [17] for definition). On $\mathbb{R}^n$, let $x = (x_1, \ldots, x_n)$ be a set of Cartesian coordinates, w.r.t. a flat metric e. Let the angular variable θ be defined by $\theta = x/r$ where $r = \| x \|_e$ is the radius.

DEFINITION 2.1. We will denote by RT^s_k the space of functions of the form

$$f(x) = \frac{f_k(\theta)}{r^k} + f_{(k+1)},$$

where $f_k \in H^{2,s}(S^n)$ and $f_{(k+1)} \in H^s_\delta(\mathbb{R}^n)$ where $\delta = \epsilon + k - n/2$ for some $\epsilon \in]0,1[$. We will take ϵ to be fixed and leave out reference to it in our notation.

If f_k has even or odd parity, then we say that $f \in RT^s_{k,e}$ and $RT^s_{k,o}$ respectively. Further, if f_k satisfies $\int_{S^n} f_k(\theta)\, d\theta = 0$, i.e. f_k is orthogonal to constants, then we say that $f \in RT^s_{k,\perp}$.

We define a norm on the RT spaces as follows:

$$\| f \|^2_{k,s} = \| f_k \|^2_s + \| f_{(k+1)} \|^2_{s,\delta}.$$

We will use the notation $f_{(j)}$ to indicate terms of order $\geqslant j$ in $1/r$. ∎

Remark. Clearly, one could define analogous function spaces with arbitrary order of radial smoothness. However, this will not be used in this paper, although the study of Einsteins equations under stronger assumptions on radial smoothness is an interesting problem in its own right, see eg. [9]. Regge and Teitelboim assumed two degrees of radial smoothness. This assumption has no important consequences for the problem studied in this paper, but the expression for the boost momentum derived in [10] (which involves dN) is more complicated than the one derived in [37] which has the same form as the ADM mass formula. ∎

The above norm makes the spaces $RT^s_k(\mathbb{R}^n)$ into Hilbert spaces. The choice of ϵ given above is important to make sure that solutions to elliptic systems of equations stay in the same class of spaces. The results on elliptic operators acting between the H^s_δ spaces carry over to the RT spaces without difficulty, although some care has to be taken due to the fact that in general, logarithmic terms appear in the solutions to Poisson's equation. These problems are eliminated by imposing the parity conditions defined below.

Let M be a C^∞ manifold, diffeomorphic to $\mathbb{R}^n$.

DEFINITION 2.2. Let e be a given Euclidean metric on M and let $r : M \to \mathbb{R}$ be defined by $r(x) = | x - x_0 |$ for some $x_0 \in M$ and for some $R \in \mathbb{R}$, let ϕ_1, ϕ_2 be a partition of unity such that $\phi_1(x) = 1$ if $r(x) \leqslant R$ and $\phi_2(x) = 1$ if $r(x) \geqslant 2R$. A Riemannian metric g on M is said to be radially smooth of order (k, m, s) if there is an $R \in \mathbb{R}$ such that

1) $\phi_1 g \in H^s(S^2 T^* M)$.

2) $$\phi_2(g-e) \in RT^s_k(S^2T^*M).$$

The space of radially smooth metrics of order (k, s) on M is denoted by $\mathbf{M}^s_k$. The space of metrics with even or odd parity is denoted in the natural way. ∎

Remark. The spaces $\mathbf{M}^s_k$ and $\mathbf{M}^s_{k,e}$ are smooth Hilbert manifolds. The inverse limits as $s \to \infty$ are strong ILH manifolds in the sense of Omori [36]. ∎

The spaces of initial data studied in [10] can be defined as follows:

DEFINITION 2.3. (The Parity Conditions). We will let $T^*\mathbf{M}_{RT}$ be given by the set of (γ, π) such that

$$\gamma \in \mathbf{M}^s_{1,e}$$

and

$$\pi \in RT^{s-1}_{2,o}(S^2TM)$$

where $s > n/2 + 2$. This means that $\gamma = c + \frac{\gamma^1(\theta)}{r} + \gamma^{(2)}$ where γ^1 is even and $\gamma^{(2)} \in H^s_\delta$ for $\delta \in]-\frac{1}{2}, \frac{1}{2}[$. Similarly, $\pi = \dfrac{\pi^1(\theta)}{r^2} + \pi^{(2)}$ where π^1 is odd and $\pi^{(2)} \in H^{s-1}_{\delta+1}$.

The corresponding concept in a four dimensional setting is the following: Let $g \in Lor^s(V)$. Then we say that $g \in Lor^s_{RT}(V)$ if there exists a slicing $i : \mathbb{R} \times \times M \to V$ such that the induced data $(\gamma_\lambda, \pi_\lambda) \in T^*\mathbf{M}^s_{RT}$ for each $\lambda \in \mathbb{R}$. ∎

See Appendix A for some remarks on analysis under the above assumptions.

Remark. (1) In the analysis $Diff_I$ below one is lead to study the operators $\Delta_{L,\gamma}$ and $\Delta_{K,\gamma}$ acting between $H^{s+1}_{\delta-1}$ and $H^{s-1}_{\delta+1}$. The choice of δ in Definition 2.3 means that the operators act as isomorphisms. See Appendix A for a discussion of related questions.

(2) In [18] a version of the H^s_δ spaces is used which is based on a Lorenz signature metric, with r replaced by a radial measure ρ defined by $\rho^2(x) = = |\eta(x, x)|$ where η denotes the Minkowski metric on $\mathbb{R}^4$. Further, in [9], radial smoothness was defined directly in terms of 4-dimensional objects, again w.r.t. a radial distance ρ. The definitions used in this paper, based on a 3 + 1 setting are easily seen to correspond to definitions using the above mentioned concepts. ∎

We define the group $Diff^{s+1}_{RT}$ to be the group of all diffeomorphisms of V which leave the set Lor^s_{RT} invariant. The following result can be found in [37,

Appendix A] and with more detail in [10, §3].

LEMMA 2.1. *The group $Diff_{RT}$ is a semidirect product $Diff_{RT} = Diff_S \ltimes P$ where P is the Poincaré group and Lie_{Diff_S} corresponds to lapses and shifts satisfying $(N, X) \in RT^s_{0,o} \times RT^s_{0,o}(TM)$. Further, $Diff_S$ is a normal subgroup of $Diff_P$ and there is a distinguished subgroup $P = Diff_{RT}/Diff_S$.* ■

Remark. The group $Diff_S$ contains no isometries. In fact, if g is not flat, then the only isometries of g under the present conditions are rotational symmetries and hence spatial transformations. ■

2.1. The York Quasi-Isotropic Gauge

The existence of supertranslations at null infinity I was first pointed out in the work of Bondi et. al. The existence of supertranslations also at spatial infinity (*Spi*) was pointed out by Bergmann [11] who presented an argument claiming to show that it is impossible to remedy the situation by «reducing» w.r.t. the supertranslation part of the gauge group. The concept of supertranslations at *Spi* has surfaced in a number of papers dealing with compactification schemes to analyze the asymptotic structure of the gravitational field [25], [6].

In the paper [6], A. Ashtekar and R.O. Hansen introduced a method for fixing the gauge freedom w.r.t. supertranslations. The basic requirement in this scheme is the vanishing of the limit at *Spi* of the magnetic part B of the Weyl tensor. It is also possible to formulate this condition in a 3 + 1-framework in terms of a condition on the radial-angular part of the Ricci curvature tensor of the 3-metric g [7] (see also [44, p. 56]).

An approach to this question, which made use of the difference in the dynamical structure at *Spi* from that at I was presented by Arnowitt [4]. The central idea here was the concept of «wave-front», i.e. that due to the finiteness of energy, it is possible to separate the dynamical parts of the field from the gauge parts near infinity, and thus, by a change of coordinates, «improve» a given set of initial data in order to satisfy stronger conditions allowing for well-defined angular momentum. This idea was taken up by J. York [43], who introduced the conditions which will be defined below.

We will here formulate the conditions corresponding to the Quasi Isotropic (QI) gauge condition of York for the RT spaces of initial data introduced above, satisfying not only asymptotic flatness in terms of H^s_δ spaces but also radial smoothness and parity conditions. Enforcing the QI conditions will allow us to fix a «frame at infinity», which will simplify the proof of the slice theorem. Further, using the QI condition with the parity conditions in force makes it possible to control the ADM momentum w.r.t. the Lie algebra of the corresponding gauge group, something which does not appear to be possible using only finite energy conditions. Finally, the extra condition on the decay of $\mathrm{tr}\pi$ is natural for proving the existence of maximal slicings (see §2.3 below).

DEFINITION 2.4. (Quasi-Isotropic Gauge): Let e be a given flat metric on M and for a two tensor w, let $\tilde{w}$ denote the trace free part of w w.r.t. e, i.e. $\tilde{w} = w - \frac{1}{3} e \,\mathrm{tr}_e w$. Let H^s_δ be the space of lower order parts of the elements $\gamma \in \mathbf{M}_{RT}$.

(1) We define the space $T^*\mathbf{M}_{QI}$ of initial data satisfying the Quasi-Isotropic gauge condition to be the set of $(\gamma, \pi) \in T^*\mathbf{M}_{RT}$ such that

$$\delta_e \tilde{h} \in H^{s-1}_{\delta+1}$$

where $h = \gamma - e$ and

$$\mathrm{tr}_e \pi \in H^{s-1}_{\delta+1}$$

(2) Let g be a Lorenz metric on V. Then we say that g is QI asymptotically flat if there is a slicing $i : M \times \mathbb{R} \to V$ such that the induced data satisfy part (1). The space of QI asymptotically flat Lorenz metrics in Lor^s_{RT} will be denoted by Lor^s_{QI} and the space of slicings $i : M \times \mathbb{R} \to V$ will be denoted by Σ_{QI}. ∎

To clarify this notion we make a few remarks. First, note that for a given flat metric e on M, there are other flat metrics in the same RT-class as e, which are not in $\mathbf{M}_{QI}$. Further, having chosen $e \in \mathbf{M}_{RT}$, we could use any $\gamma \in \mathbf{M}_{QI}$ instead of e in the definition. The QI condition should be viewed as a slice condition which fixes part of the gauge freedom at infinity. To see what this means, let $h \in T_e\mathbf{M}_{RT}$ and consider the York decomposition of h w.r.t. e (see [43] or Appendix A.3 for notation):

$$h = h_{TT} + h_T + L_e(W).$$

Then, the QI condition becomes $\Delta_{L,e} W \in H^{s-1}_{\delta+1}$, i.e. the only constraint is on the longitudinal part $L_e(W)$ of h. In general, for $h \in T_\gamma\mathbf{M}_{RT}$, $W = O(1) + H^{s+1}_{\delta-1}$. Thus, what the condition states is that the $O(1)$ part of W vanishes. Similarly, the QI condition forces the $O(1/r^2)$ part of π_T to vanish. Here π_T denotes the trace part of π in the decomposition $\pi = \pi_{TT} + \pi_T + L_e(Y)$.

In the paper [4] it was claimed that given «general» initial data with finite energy, it is possible to find a coordinate system in which the initial data satisfies conditions of the type given by Definition 2.4. This will not be done here but it should not be difficult to prove the existence of QI coordinates for general asymptotically flat metrics using techniques similar to those used in proving the existence of harmonic coordinates near infinity, see eg. [35].

Remark (Linearization Stability). The space $T^*\mathbf{M}^s_{QI}$ will be the phase space for GR in the rest of this work and the constraint set $\mathbf{C}^s_{QI} = \{(\gamma, \pi) \in T^*\mathbf{M}^s_{QI} \mid \Phi(\gamma, \pi) = 0\}$ will be used to describe the set of asymptotically flat solutions of Einsteins equations. The well known fact that $T^*\mathbf{M}^s$ is a smooth manifold extends easily to the present situation and the limit as $s \to \infty$ yields an ILH structure on $T^*\mathbf{M}^s_{QI}$.

Similarly, we expect, due to the nonexistence of Killing fields in Lie_{Diff_I} that $\mathbf{C}^s_{QI}$ should be a smooth manifold. However, in the case of $\mathbf{C}^s_{QI}$, the situation is more subtle. The only proofs of the smoothness of the constraint set in the asymptotically flat case use the assumption that $\mathrm{tr}\pi = 0$, i.e. the maximal slicing condition (see [16]). Therefore, assuming the existence of maximal slices (see §2.3 below), it would be possible to complete the proof. The asymptotic conditions used here do not change the situation.

In the paper by Beig and O-Murchadha [10], there is an argument towards proving the linearization stability for the constraint equations for GR. The basic idea is to prove that $\ker D\Phi^* = 0$ and then appealing to the nondegeneracy of the symplectic form, instead of proving directly that $D\Phi$ is surjective under appropriate conditions.

The argument in [10], seems to be able to handle $\mathrm{tr}\pi \neq 0$ but to complete the proof, some further analysis is needed to be able to appeal to the inverse function theorem, since the operator which is studied ($D\Phi^*$) is the formal adjoint to the operator which is of interest ($D\Phi$). Also, the relation between the range of Φ and the domain of definition of $D\Phi^*$ is not completely straightforward, so some care is required to complete the argument.

The following result characterizes the group of diffeomorphisms which is picked out by the QI condition.

THEOREM 2.2. (the group $Diff_P$). *Let the space $Ein^s_{QI}(V)$ be as above and let $Diff^{s+1}_P$ denote the space of all diffeomorphisms of V which leaves $Ein^s_{QI}(V)$ invariant. Then*

$$Diff^{s+1}_P(V) \cong Diff^{s+1}_I \ltimes P,$$

where $Diff^{s+1}_I$ denotes the space of those diffeomorphisms corresponding to lapses and shifts (N, X) satisfying

$$(N, X) \in H^{s+1}_{\delta-1}(\mathrm{F} \times TM)$$

Proof. From Lemma 2.1 recall that the group which leaves the parity conditions invariant is of the form

$$Diff_{RT} = Diff_S \circledS P,$$

where $Diff_S$ denotes the diffeomorphisms asymptotic to supertranslations (in the sense of Regge and Teitelboim). Thus the problem here is to show what the effect is of imposing the QI conditions.

Let $(\gamma, \pi) \in T^*\mathbf{M}_{QI}$ and assume that $(N, X) \in Lie_{Diff_S}$. By Lemma 2.1 this means that we can write $N = N_{-1} + N_{(0)}$ and $X = X_{-1} + X_{(0)}$, where (N_{-1}, X_{-1}) corresponds in a unique way to an element of Lie_P and $(N_{(0)}, X_{(0)}) \in RT^s_{0,o} \times \times RT^s_{0,o}(TM)$, i.e. $(N_{(0)}, X_{(0)})$ is an infinitesimal supertranslation in the sense of Regge and Teitelboim.

Then assuming $(N, X) \in Lie_{\,Diff_P}$ means that $\mathbf{J} \circ D\Phi^*(N, X) \in T_{(\gamma,\pi)}(T^*\mathbf{M}_{QI})$. Hence $\mathbf{J} \circ D\Phi^*$ denotes the generator of the dynamics of GR in the $3 + 1$ picture, see [28].

Applying Definition 2.4 and using the explicit form of $\mathbf{J} \circ D\Phi^*$ gives the condition of the form

$$\delta_\gamma(N\tilde{\pi}) + \Delta_{L,\gamma} X \in H^{s-1}_{\delta+1}$$
$$\Delta_\gamma N + \delta_\gamma(X\pi) \in H^{s-1}_{\delta+1}.$$

Here $\tilde{\pi}$ denotes the trace free part of π w.r.t. e. By the arguments used to prove Lemma 2.1 (cf. [37, Appendix A] and [10, §3]) we find by setting $X = 0$ that $\delta_\gamma(N\tilde{\pi}) \in RT^{s-1}_{2,o}$ and by setting $N = 0$ that $\delta_\gamma(X\pi) \in RT^{s-1}_{2,o}$. Further, one sees that the only part of (N, X) which contributes to the $1/r^2$-parts is the infinitesimal Poincaré transformation (N_{-1}, X_{-1}).

Decomposing the terms $\Delta_{L,\gamma} X \in H^{s-1}_{\delta+1}$ and $\Delta_\gamma N$ into pieces corresponding to (N_{-1}, X_{-1}) and lower order parts gives after some rearrangements the following condition near infinity:

(2.1-a) $\quad \Delta_e N_{(0)} = F_2(\theta)/r^2 + l.o.$

(2.1-b) $\quad \Delta_{L,e} X_{(0)} = G_2(\theta)/r^2 + l.o.$

The R.H.S. is a function of (N_{-1}, X_{-1}) and (γ, π) and is odd. Clearly the equations (2.1) can be solved near infinity to give an expression for the $O(1)$ supertranslation part of (N, X) in terms of (N_{-1}, X_{-1}) and other data. ■

Remark. It should be noted that the finite energy conditions [35] or the conditions used in [18] do not allow a result like Theorem 2.2. Therefore some stronger condition like the parity condition has to be used as a starting point. It would be interesting to know whether the more general conditions, which still imply finite angular momentum, introduced by Chruschiel [20], would allow a proof of a result like the above. ■

2.2. The Frame at infinity

By a well known construction (see e.g. [9]) it is possible, under the present asymptotic conditions, to complete V in the spatial direction by adding a 'hyperboloid at infinity' $\mathbf{H}$. The space $\mathbf{H}$ is isomorphic to the unit hyperboloid $\mathbf{H}_0$ in Minkowski space $\mathbb{R}^4$.

Let $\Sigma_{\mathbf{H}}$ denote the space of cross sections of $\mathbf{H}$, corresponding to the set of intersections of $\mathbf{H}_0$ by spatial hyperplanes in $\mathbb{R}^4$, which we denote by $\Sigma_{\mathbf{H}_0}$. Denote by $P_{\mathbf{H}_0}$ the group of automorphisms of $\Sigma_{\mathbf{H}_0}$ induced by the action of the Poincaré group P on $\mathbb{R}^4$. We will denote by $[A]$, the element in $P_{\mathbf{H}_0}$ corresponding to $A \in P$.

The following result is an easy consequence of the construction of $\mathbf{H}$.

LEMMA 2.4. *There is a well defined map of Σ_{QI} onto $\Sigma_{\mathbf{H}}$, defined by taking limits as $r \to \infty$. We denote the image of $i \in \Sigma_{QI}$ by $[i]$. The action of $Diff_P$ on V induces an action on $\Sigma_{\mathbf{H}}$. For $\eta \in Diff_P$, we denote the corresponding element in $P_{\mathbf{H}}$ by $[\eta]$. Further, if $\eta \in Diff_I$, then $[\eta] = id \in P_{\mathbf{H}}$.* ■

2.3. Maximal Slicings

The general existence of maximal and CMC slicings is still an open question, and as in the case of CMC slicings in the spatially compact case, we will have to

CONJECTURE (Existence of Maximal Slicings). *Let $V = \mathbb{R}^3 \times \mathbb{R}^1$ and let $g \in Ein^s_{QI}(V)$. Then, for any slicing $i \in \Sigma_{QI}$ there exists a unique slicing $j \in \Sigma_{QI}$ such that $[j] = [i]$ and $j_\lambda(M)$ is a maximal hypersurface of (V, g) for each $\lambda \in \mathbb{R}$.* ■

The work of Bartnick [8], which proves the existence of maximal slicings asymptotic to regular time functions under the assumption that certain bounds hold in the interior of V, uses asymptotic conditions with $\operatorname{tr}\pi = O(\frac{1}{r^3})$. Therefore, the QI conditions are natural in this context.

D. Witt [41] recently pointed out the existence of topological obstructions to the existence of maximal initial data. The obstruction results from the fact that the condition $f_r\pi = 0$ together with the maximal constraint implies $R(\gamma) \geqslant 0$. On the other hand, there do exist solutions to the constraint equations for spacelike slices with arbitrary topology, both for the compact and asymptotically flat cases.

The existence of topological obstructions has as a consequence that a number of results where the existence of maximal slicings has been assumed need new proofs for general spatial topologies. Examples of this is the proof of smoothness of the constraint set [16] and the characterization of the critical points of the mass function [15].

In the case where $M = \mathbb{R}^3$, then it known [13], [15] that the only critical point of the mass function is flat space. The fact that the mass function has a unique critical point indicates that the space of solutions to Einsteins equations is connected in the case $M = \mathbb{R}^3$, This problem is discussed in the work of O-Murchadha [33]. What is lacking for a complete proof is the result that the mass function for GR has a critical point in every component of the constraint set, which seems to be hard to prove. The connectedness of the solution set would together with the stability of maximal hypersurfaces under deformations [14] imply the global existence of maximal hypersurfaces.

2.4. The Isometry groups

For $g \in Lor(V)$, let I_g denote the group of isometries of g. Assume g is not flat and that g has a time like Killing field. Then, in some rest frame the mass is zero which by the Positive Mass Theorem is a contradiction. This proves the following result.

LEMMA 2.5. *Let $g \in Ein^s_{QI}$ and assume that g is not flat. Then g has no time like killing fields.* ∎

Thus, if g is not flat, any $\eta \in I_g$ is spacelike and it is not hard to see that this implies that there is a frame $l \in \Sigma_{\mathrm{H}}$ which is invariant under the action of $[\eta]$. If we let j denote the maximal slicing corresponding to l, then the uniqueness of maximal slicings implies that η induces a diffeomorphism η_t of M such that $\eta \circ j_t = j_t \circ \eta_t$. If we let (γ_t, π_t) denote the data induced on $M \times \{t\}$, we see that η_t is an isometry of γ_t.

On the other hand, if η is a timelike isometry, then g is flat by Lemma 2.5 and V is isometric to Minkowsky space. In this case η corresponds to a timelike Poincaré transformation, i.e. a combination of timelike translation and boost.

The following result is the asymptotically flat version of [24, Proposition 2.3].

THEOREM 2.6. (Structure of data for isometries). *Assume that $g \in Ein^\infty_{QI}(V)$ has a nontrivial Killing field Z. For a slicing j, let $(\gamma_\lambda, \pi_\lambda)$ denote the induced data on $j_\lambda(M)$ and let (N, X) denote the lapse and shift corresponding to Z.*

One of the following statements is true:

1) Z is time like. Then g is flat and V is isometric to Minkowski space. Hence there exists a maximal slicing j such that the induced data $(\gamma_\lambda, \pi_\lambda)$ is such that γ_λ is flat and $\pi_\lambda \equiv 0$.

2) Z is spacelike. Let j denote an adapted maximal slicing for Z which exists by the above arguments. Then in terms of j, the lapse N vanishes the data $(\gamma_\lambda, \pi_\lambda)$ satisfy $L_X \gamma_\lambda = 0$ and $L_X \pi_\lambda = 0$ for all $\lambda \in \mathbb{R}$. ∎

Using the corresponding results for the asymptotically flat Riemannian case [1, §3], we can now prove:

PROPOSITION 2.7. *If g is not flat, then I_g is a compact Lie group, isomorphic to a subgroup of $O(3)$, the 3-dimensional orthogonal group, while if g is flat, then $I_g = P$, the Poincaré group. The map $i : I_g \to Diff_P^s$ is an imbedding, the quotient $Diff_P^s/I_g$ is a manifold and the map $\pi : Diff_P^s \to Diff_P^s/I_g$ admits smooth cross sections.* ■

3. PROOF OF THE SLICE THEOREM

In this section, we will give a proof of the slice theorem, Theorem 1.1, for the action of $Diff_P(V)$ on the space $Ein_{QI}(V)$ of asymptotically flat solutions of Einsteins equations satisfying the QI conditions. This is a generalization of the results proved in [28] and [1].

The proof of the slice theorem consists of two main steps. The first is differential topological in nature and consists of showing that the orbits are immersed submanifolds and that a slice exists for the action of $Diff_P$.

The proof that the orbits are immersed submanifolds proceeds exactly as in [28, §5]. The central step of the proof here is using the ellipticity of the 3 + 1 form of the mapping $X \to \delta_g(L_X g)$ to imply the closed range property, which allows the use of standard techniques from differential topology. The construction of the slice is an easy extension of techniques used in [28] and [1] and will not be discussed here.

The second step consists of proving that the orbit $\mathbf{O}^s(g)$ is closed in the topology induced from $T^*\mathbf{M}_{QI}^s$. Here one uses in the compact case the (conjectured) fact that there is a globally unique constant mean curvature (CMC) slicing for any g satisfying reasonable hypotheses. The reason that the procedure used in [23] and [1] does not work is that the set of ON-frames at a point for a given Lorenzian metric is noncompact. The existence and uniqueness of CMC slicings allows one to separate out spatial directions and apply the method for the Riemannian case there.

In the noncompact case there is no uniqueness of CMC slicings. Instead, one has for a given value τ of the mean curvature, existence and uniqueness results, given a «slicing at infinity». In this paper we are working in a spatial infinity context and therefore replace the CMC slicings used in [28] by maximal slicings.

A question which is closely related to the global existence of maximal and CMC slicings and which is still open, is the global existence (in time) of solutions of

Einsteins' equations. The work of Christodoulou and O-Murchadha [18] shows quasiglobal existence and in particular that the maximal development of an asymptotically flat solution of Einsteins' equations admits boosts. This work was done in terms of weighted Sobolev spaces, but carries over to the present setting without difficulty. In [19], Christodoulou shows global existence for spherically symmetric solutions.

3.1. Proof that orbits are closed submanifolds

We will only consider the case of $Diff_P$, the case of $Diff_I$ being similar and easier. The strategy for the proof that the orbits are closed is the following: First, we note that under the QI asymptotic conditions used here, the action of $Diff_P$ induces an action on the set of cross sections of spatial infinity, a locally compact set (this is Lemma 2.4). Next we show that the assumption of nonzero mass implies that there is a convergent subsequence in the sequence of cross sections induced by the action of any sequence of diffeomorphisms η_n such that $\eta_n^* g_0$ converges for some $g_0 \in Lor_{QI}(V)$. The existence of a maximal slicing corresponding to the limiting cross section enables us to control the behaviour of the η_n.

Let $g_0 \in Ein^s_{QI}(V)$ be given. To show that $\mathbf{O}^s(g_0)$ is a closed subset of $Ein^s_{QI}(V)$, we proceed as follows. Consider a sequence $\{\eta_n\}_{n=1}^{\infty} \subset Diff_P^{s+1}$ and assume that $g_n = \eta_n^* g_0$ has the property that there exists a $g_\infty \in Ein_{QI}(V)$ such that

$$\lim_{n \to \infty} g_n = g_\infty .$$

Then we will prove

THEOREM 3.1. *There exists a* $\eta_\infty \in Diff_P^{s+1}$ *such that* $g_\infty = \eta_\infty^* g_0$. ■

Assume that g_0 is not flat. The case where g is flat is similar to the corresponding spatially compact case and is left to the reader. Let i^0 be a given slicing of V such that g_0 satisfies the QI conditions w.r.t. i^0. Let $[i^0]$ denote the cross section of $\mathbf{H}$ defined by i^0 and let $l_n = [\eta_n] \in P_{\mathbf{H}}$ denote the corresponding action of η_n (see §2.2 above for these concepts).

We will now prove that if g is not flat, then $\{l_n[i^0]\}$ has a convergent subsequence. The following Lemma is a preliminary result which will allow us to control the behaviour at infinity of g_n. The idea of proof is similar to that used in the proof of the corresponding statement for the asymptotically Euclidean case in [1].

LEMMA 3.2. *Let* r *be a radius function determined by geodesic distance w.r.t.* g_0

and let ρ be a luminosity distance in V. Assume that g_0 is not flat. Let $p \in V$ and let $p_n = \eta_n(p)$. Then the sequences $r(p_n)$ and $\rho(p_n)$ are both bounded.

Proof. Assume that either $r(p_n) \to \infty$ or $\rho(p_n) \to \infty$ as $n \to \infty$. The asymptotic flatness of g_0 now implies that $\Gamma(p_n)$ and $\nabla\Gamma(p_n)$ tend to 0 as so we can find geodesic neighborhoods U_n of p_n which become arbitrarily large as $n \to \infty$.

Now consider the sequence $W_n = \eta_n^{-1}(U_n)$ of neighborhoods of p. From the above we see that as $n \to \infty$, the neighborhoods W_n grow to cover V and that by choosing n large enough we see that g_n is arbitrarily close to a flat metric on any compact set. But by assumption, $g_n \to g_\infty$ and so we find that g_∞ must be flat. We will now show that this is a contradiction.

The Positive Mass Theorem [38] tells us that if P_μ is the ADM momentum vector, then $|P(g)| > 0$ for any $g \in Ein(V)$ which is not flat, where $|\,|$ denotes the invariant semi-norm on $\mathfrak{t}^*$, the dual of the space of translations, given by the Minkowski metric. In particular, $P_\mu(g) = 0$ if and only if g is flat. By the transformation properties of P_μ [37], $|P(g_n)| = |P(g_0)|$. Further, we know that the mapping $P : Ein(V) \to \mathfrak{t}^*$ is continuous, which implies that $P(g_\infty) = P(g_0) \neq 0$. But the argument above shows that $r(p_n) \to \infty$ or $\rho(p_n) \to \infty$ implies that g_∞ is flat and hence $P(g_\infty) = 0$, so we have derived a contradiction. ∎

The Lemma enables us to control the behaviour of η near infinity. Let $l_n = [\eta_n] \in P_{\mathrm{H}}$ as above.

PROPOSITION 3.3. *If g is not flat, then there exists a $l_\infty \in P_{\mathrm{H}}$ which is the limit of a subsequence $\{l_k\}$ of $\{l_n\}$.*

Proof. Assume $l_n \to \infty$ in P_{H}. First consider the case where the boost part of $l_n \to \infty$. Then, for $p \in V$ which is «close to infinity», $l_n \to \infty$ means that $\eta_n(p)$ moves to infinity in V approximately along a hyperboloid of r = constant. But by Lemma 3.2, this contradicts the assumption that g is not flat and we can thus assume that the boost part of l_n stays bounded.

In fact, for any frame $[i^0]$, we can choose a frame $[j]$ such that a subsequence of $[i^n] = l_n[i^0]$ converges to pure time translations of $[j]$, i.e. $[i_t^n] \to [j_{t+\lambda_n}]$ for some sequence $\{\lambda_n\} \subset \mathbb{R}$. From now on we assume that such a subsequence has been chosen.

Let i^n be the maximal slicing of g_n w.r.t. $[i^n]$ and let j^n be the maximal slicing of g_n w.r.t. $[j]$. Let the data induced by i^n on $M \times \{t\}$ be denoted by $\gamma_n(t)$, $\pi_n(t)$. Similarly, let the data induced by j^n be denoted by $\tilde{\gamma}_n(t)$, $\tilde{\pi}_n(t)$. Then an argument similar to the proof of [28, Proposition 3.4] shows that there exists diffeomorphisms $\chi_{t,n} \in Diff_P^{s+1}(M)$ such that $i_t^n \circ \chi_{t,n} \to j_{t+\lambda_n}^n$ as $n \to \infty$. If we fix $t = 0$ then we can rewrite this as

$$i_{t_n}^n \circ \chi_n \to j_0^\infty,$$

with $t_n = -\lambda_n$ and $\chi_n = \chi_{t_n,n}$. Further we find that as $n \to \infty$

$$(\chi_n^* \gamma_n(t_n), \chi_n^* \pi_n(t_n)) \to (\tilde{\gamma}_\infty(0), \tilde{\pi}_\infty(0)).$$

The uniqueness of maximal slicings implies that $\eta_n \circ i_t^n = i_t{}^0 \zeta_n(t)$ which means that $\gamma_n(t) = \zeta_n^*(t)\gamma_0(t)$. After restricting to t_n we get

$$\eta_n \circ i_{t_n}^n = i_t{}^0 \zeta_n$$

where we use the notation $\zeta_n = \zeta_n(t_n)$. This means that $\gamma_n(t_n) = \zeta_n^* \gamma_0(t_n)$. Thus we find that as $n \to \infty$,

$$\tilde{\eta}_n^* \gamma_0(t_n) \to \tilde{\gamma}_\infty(0)$$

where $\tilde{\eta}_n = \zeta_n \circ \chi_n$.

Now assume that $t_n \to \pm\infty$. By the time reversability of Einsteins equations, we may assume that $t_n \to +\infty$. Consider the sequence $\gamma_0(t_n)$. As $t \to \infty$ two things can happen: either the gravitational energy disperses to null infinity, in which case $\gamma_0(t_n)$ tends to a flat metric, or a black hole forms, in which case the curvature of $\gamma_0(t_n)$ grows unbounded. Both of these possibilities are ruled out by the above.

In the first case we find that $\gamma_\infty(0)$ must be flat. But the choice of 0 as reference point is arbitrary and hence g_∞ must be flat. This is a contradiction by the same argument as in the proof of Lemma 3.2. In the second case we find that $\gamma_\infty(0)$ has unbounded curvature, which is ruled out by our assumptions. Thus t_n must be bounded. This completes the proof of Proposition 3.3. ■

In the following we will restrict our attention to a subsequence η_n given by the Proposition, such that $[\eta_n]$ converges. It follows from the Proposition that $[i^n] = l_n[i^0]$ converges to some frame $[i^\infty]$. By assumption (§2.3) there is a unique maximal slicing $j^n : M \times \mathbb{R} \to V$ defined by $[i^n]$ and g_n.

Let the data induced by j^n on $M \times \{t\}$ be denoted by $\gamma_n(t)$, $\pi_n(t)$. Then an argument similar to the proof of [28, Proposition 3.4] tells us that there exists diffeomorphisms $\chi_{t,n} \in Diff_P^{s+1}(M)$ such that

$$j_t^n \circ \chi_{t,n} \to j_t^\infty$$

and

$$(3.1) \qquad (\chi_{t,n}^* \gamma_n(t), \chi_{t,n}^* \pi_n(t)) \to (\gamma_\infty(t), \pi_\infty(t)).$$

Here we use $Diff_P^{s+1}(M)$ to denote the spatial part of $Diff_P^{s+1}(V)$. We can now complete the proof in a way analogous to that in [28]. The uniqueness of maximal slicings means that if i_t is a maximal slicing for g and $\eta \in Diff_P^{s+1}$, then if j_t is a maximal slicing w.r.t. $(\eta^*g, [\eta * i])$ there exists a diffeomorphism $\zeta(t) \in Diff_P^{s+1}(M)$ such that $\eta \circ j_t = i_t \circ \zeta(t)$. Thus, there exists a sequence $\zeta_n(t) \in Diff_P^{s+1}(M)$ such that

$$(3.2) \qquad \gamma_n(t) = \zeta_n(t)^* \gamma_0(t)$$

for each t. Now (3.1) and (3.2) gives us

$$\lim_{n \to \infty} \eta_n^*(t)\gamma_0(t) = \gamma_\infty(t),$$

where $\eta_n(t) := \chi_n(t) \circ \zeta_n(t) \in Diff_P^{s+1}(M)$. We can now apply [1, Theorem 3.5] (see also Appendix B) to show that for each $t \in \mathbb{R}$, the sequence $\eta_n(t)$ has a covergent subsequence. Thus, for every $t \in \mathbb{R}$, we find a$\{\eta_\infty(t) \in Diff_P^{s+1}(M)$ such that

$$\eta_\infty(t)^* \gamma_0(t) = \gamma_\infty(t)$$

and the condition

$$\eta_\infty \circ i_t^\infty = i_t^0 \circ \eta_n(t)$$

defines a unique element of $Diff_P^{s+1}$ such that η_∞ is a limit of a subsequence of η_n.

The proof that $\mathbf{O}^s(g)$ is an embedded submanfold can now be completed by a standard topological argument, see [28, Lemma 5.2].

Appendix A: Radial Smoothness

We will recall some elementary facts about functions defined in terms of power series in a radial parameter. On $\mathbb{R}^3$, let $x = (x_1, x_2, x_3)$ be a set of Cartesian coordinates, w.r.t. a flat metric e. Let the angular variable θ be defined by $\theta = x/r$ where $r = \| x \|_e$ is the radius.

If we introduce radial coordinates (r, θ) on $\mathbb{R}^3$, then the Laplace operator becomes

$$\Delta_e = \left[\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_\theta \right]$$

Therefore, if $f(x) = \dfrac{g(\theta)}{r^k}$ we have that on $\mathbb{R}^{3*} = \mathbb{R}^3 - \{0\}$, $\Delta_e f = \dfrac{1}{r^{k+2}}$ ·

$(\Delta_\theta + c_k)g(\theta)$, where $c_k = k^2 - k$.

The space of solutions to $\Delta_e f = 0$ on $\mathbb{R}^{3*}$ for f of the form $f(x) = \frac{g(\theta)}{r^k}$ is spanned by elements of the form $r^k Y_k$ and $r^{-(k+1)} Y_k$ where Y_k is a spherical harmonic of order k, i.e. a solution to

$$\Delta_\theta Y_k + k(k+1)Y_k = 0.$$

See [40, p. 303] or any textbook on PDE in mathematical physics for details.

This implies in particular that for $k \in \mathbb{R}$, the only solutions on $\mathbb{R}^{3*}$ to the equation $\Delta_e f(x) = \frac{1}{r^k}$ are $f(x) = \frac{c}{r^{k-2}}$ for $k \neq 2$ and $k \neq 3$. In the exceptional cases we have $f(x) = c\ ln(r)$ for $k = 2$ and $f(x) = c\ \frac{ln(r)}{r}$ for $k = 3$ for some constants c.

A.2. Poisson's Equation with Radial Smoothness

Recall that if $f : \mathbb{R}^3 \to \mathbb{R}$ is integrable, then the solution to $\Delta_e u = f$ can be written as $F{*}f$ where $F(x) = cr^{-1}$ for some constant c. This implies that when $f \in H^s_\delta$ for $\delta > 3/2$ (i.e. $f = o(\frac{1}{r^{3+\epsilon}})$) we have that $u = \frac{M}{r} + g$ where M is a constant and $g \in H^{s+2}_{\delta-2}$. Under stronger assumptions of f, it is possible to give detailed information on the form of the leading terms in u to higher order in $1/r$.

Next consider the case where $f = O(\frac{1}{r^3})$, i.e. f may fail to be integrable. Due to the fact that $ker\ \Delta_\theta = coker\ \Delta_\theta$ = constants, the function $\frac{1}{r^3} + l.o.$ is not in the range of $\Delta_e : RT^s_1 \to RT^{s-2}_3$. We see that $\Delta_e : RT^s_{1,\perp} \to RT^{s-2}_{3,\perp}$ is an isomorphism. Finally, we will consider the equation

$$\Delta_g u = f$$

for radially smooth g. As an example, let f be of the form

$$f = \frac{f_1}{r^3} + \frac{f_2}{r^4} + l.o.$$

Then, assuming that g is sufficiently radially smooth, it is readily shown that u will be of the form

$$u = a\frac{lnr}{r} + \frac{h_1}{r} + bY_1\frac{lnr}{r^2} + \frac{h_2}{r^2} + l.o.$$

where a and b are constants and Y_1 is a first order harmonic.

A.3. The Basic PDO and Their Properties

In this section we will consider the properties of the operators $\Delta_{K,g}$ and $\Delta_{L,g}$ acting between spaces of radially smooth vectorfields. Since $n = 3$ is the case of interest in GR, we will restrict our attention to this.

For $X \in \Gamma(TM)$ and $g \in \mathbf{M}$, let $K_g X = L_X g$, let L_g denote the operator defined by $X \to K_g X - \frac{1}{3}\operatorname{tr}(K_g X) g$ and let δ_g denote the divergence w.r.t. g, i.e. the formal adjoint to K_g. Then we denote $\delta_g \circ K_g : \Gamma(TM) - \Gamma(TM)$ by $\Delta_{K,g}$ and $\delta_g \circ L_g$ by $\Delta_{L,g}$. The operators $\Delta_{K,g}$ and $\Delta_{L,g}$ are the «vector» Laplacians which are of importance in the study of the action of the group of diffeomorphisms and the group of conformal transformations.

The properties of the operators $\Delta_{K,g}$, $\Delta_{L,g}$ and Δ_g on asymptotically Euclidean manifolds have been studied by Christodoulou and O-Murchadha [18]. The mapping properties of the operators acting in the category of Sobolev spaces and weighted Sobolev spaces has been studied previously, and from these results it is not hard to derive the corresponding properties for the space RT^s_k.

The equations which are most interesting in our applications are of the form $\Delta_{K,g} X = \delta_g h$ for $h \in T_g \mathbf{M}^s_{RT}$, i.e. $\delta_g h \in RT^{s-1}_{2,o}$. Let $s \geqslant 2$ and on $\mathbb{R}^3$, let A_e denote one of the operators $\Delta_{K,e}$ or $\Delta_{L,e}$ and let A_g denote $\Delta_{K,g}$ or $\Delta_{L,g}$. The following result is easily proved using techniques similar to those used in [1, §2.4].

PROPOSITION A. Assume that $g \in \mathbf{M}^s_{RT}$ and let m, s be as in Definition 2.3. For $k \in \mathbf{Z}$ consider the mapping

$$A_g : RT^{s+1}_k(TM) \to RT^{s-1}_{k+2}(TM).$$

The following statements are true.

1) If $k = 1$, then A_g is an injection with range given by $RT^{s-1}_{3,\perp}(TM)$.

2) If $k = 0$ then $A_g : RT^{s+1}_{0,o}(TM) \to RT^{s-1}_{2,o}(TM)$ is an isomorphism. In the general case, without the assumption of odd leading terms, we have that A_g has a kernel consisting of the asymptotically constant vectorfields and a range $RT^{s-1}_{2,\perp}$ where the $\perp$ indicates that similar to the scalar case, the $1/r^2$-part consist of $X_2(\theta)/r^2$ where X_2 is orthogonal to the constants. If $X \in \ker A_g \cap RT^{s+1}_0$ then

$$X = X_0 + X_1,$$

where X_0 is constant and $X_{(1)} \in H^{s+1}_\delta$ with $\delta = \epsilon - 3/2$ as in Definition 2.1.

3) If $k = -1$ then A_g has kernel consisting of asymptotically 1 : st order vectorfields: if $X \in \ker A_g \cap RT^{s+1}_{-1}$ then

$$X = X_{-1} + X_{(0)},$$

where X_{-1} is a first order vectorfield in $\ker A_e \cap RT^{s+1}_{-1}$ and $X_{(0)} \in RT^{m,s+1}_{0,o}$.

If $g \in C^\infty_{loc}$, then in the above cases, $ker\, A_g \subset C^\infty_{loc}$. ■

In the case of $\Delta_{L,g}$, the form of the leading terms of solutions of $\Delta_{L,g} X = Y$ can be derived from the parametrix for $\Delta_{L,e}$ which can be found in [43].

Appendix B: The group D(M) and the space M/D.

Let $M = \mathbb{R}^3$. We will in this appendix consider the group $\mathbf{D}^{s+1}(M)$ of all diffeomorphisms of M which leave the space $\mathbf{M}^s_{RT}$ of asymptotically radially smooth Riemannian metrics with even parity invariant. The main ideas are the same as in [1] and we concentrate here on the technical points connected with the assumption of radial smoothness.

The first fact to note is that in general, for vectorfields $X^i \in H^{s+1}_{loc}$ of the form

$$X^i = c^i lnr + X^i_0(\theta) + l.o.,$$

i.e. the logarithmic translations first noted to be of importance in GR by Bergmann [11], we have that $K_g(X) \in RT^s_1$. In particular, the term $K_e(c^i lnr)$ falls off like $1/r$ and has odd parity. Hence without the parity condition, $\mathbf{D}$ would contain trasformations with a logarithmic term. It has been noted that the occurrence of such terms makes the definition of momenta for GR problematic and various schemes have been proposed to deal with this problem, see for example [5].

We see from the above that the parity condition on $\mathbf{M}$ excludes the logaritmic terms from $\mathbf{D}$. The question of importance in the study of momenta for GR is the exact form of the leading terms of elements in Lie_{Diff_P}. In this section we are concerned with the spatial part $\mathbf{D}$ of $Diff_P$. To be able to control the behaviour of $\mathbf{D}$, we use the properties of the operator $\Delta_{K,g}$. Using Proposition A as in [1, §2.4], we find that $Lie_{\mathbf{D}}$ is given by vectorfields of the form

$$X(x) = A(x) + X_1(x),$$

where $A \in \mathbf{e}$, the Lie algebra of the Euclidean group E in $\mathbb{R}^3$ and $X_1 \in RT^s_{0,o}$. The vectorfields in $RT^s_{0,o}$ which do not vanish at infinity were called supertranslations by Regge and Teitelboim and are related to the supertranslations occurring in the Spi theory [7] although here the set of supertranslations is much smaller.

PROPOSITION B. 1) *Let* $\mathbf{D}^{s+1}$ *denote the group of all diffeomorphisms which leave* $\mathbf{M}^s_{RT}$ *invariant. Then*

$$\mathbf{D}^{s+1} = D^{s+1} \circledS E,$$

where D^{s+1} *denotes the space of all diffeomorphisms* η *such that* $\eta - id \in RT^{s+1}_{0,o}$.

2) *The group* D^{s+1} *contains no isometries.* ■

Part (2) of the above Proposition is a direct consequence of Proposition A (2).

Remark. The group **D** and D are topological groups but not Lie groups and the action of $\mathbf{D}^{s+1}_{k-1,o}$ and $D^{s+1}_{k-1,o}$ on $\mathbf{M}^s_{k,e}$ is continuous but not differentiable. However, the inverse limits as $s \to \infty$, denoted by $\mathbf{D}^\infty$ and D^∞ are ILH Lie groups and the actions on $\mathbf{M}^\infty$ are ILH actions. We will not discuss these concepts in detail here, but refer the reader to the original literature [36] and [12]. See also [27]. ■

There is an important difference between the structure of the group **D** using the present asymptotic conditions and the structure of the corresponding group with asymptotic conditions defined in terms of weighted Sobolev spaces. In the case studied in [1], the group **D** is for $n = 3$ and values of δ suitable for GR, of the form $\mathbf{D} = D \circledS O$, where O denotes the rotation group in $\mathbf{R}^3$ and D denotes a group of diffeomorphisms such that roughly $\eta - id = r^{1/2-\epsilon}$ for some $\epsilon > 0$, i.e. D contains the translations. Further, as pointed out already by Bergmann for the case of GR [11], it does not appear possible in this case to select out a distinguished Euclidean subgroup, and in the 4-dimensional case a distinguished Poincaré subgroup, which is of importance for the definition of angular momenta. This type of problem has been extensively discussed in the literature, see for example [7].

Now we consider the action of **D** on **M**. One of the goals of this paper is to construct the quotient spaces $\mathbf{C}/Diff_I$ and $\mathbf{C}/Diff_P$, where **C** denotes the constraint set in the Hamiltonian formulation of GR. An important step in the construction of $\mathbf{C}/Diff_I$ is the study of the quotient spaces **M**/**D** and $\mathbf{M}/D$. This problem was studied for the asymptotically Euclidean case in [1], using weighted Sobolev spaces. The analysis is similar for the case of radially smooth metrics satisfying the parity condition.

We review the main concepts involved in constructing the quotients $\mathbf{M}^\infty/\mathbf{D}^\infty$ and $\mathbf{M}^\infty/D^\infty$ in the rest of this section. For details in the H^s_δ case, the reader is referred to [1]. Let $A : \mathbf{D} \times \mathbf{M} \to \mathbf{M}$ denote the action (by pullback) of **D** on **M**. To construct the quotient we need to prove

(1) That the orbits $A(\mathbf{D}, \gamma)$ are closed submanifolds of **M**. This involves two steps:

(1.1) Proof that the orbit is an immersed submanifold.

(1.2) Proof that the orbit is a closed subset of **M**.

(2) Construction of a (local) slice for the action. Let I_γ denote the isotropy group for $\gamma \in \mathbf{M}$. A slice for the action of **D** or D is a submanifold $S \subset \mathbf{M}$ such that the following holds:

(2.1) If $\eta \in I_\gamma$, then $A(\eta, S) = S$.

(2.2) If $\eta \in \mathbf{D}$: such that $A(\eta, S) \cap S \neq \emptyset$, then $\eta \in I_\gamma$.

(2.3) There exists a local cross section $\chi : \mathbf{D}/I_\gamma \to \mathbf{D}$ defined in a neighborhood U of the identity coset such that if $F : U \times S \to \mathbf{M}$ is defined by $(u, t) \to A(\chi(u), t)$, then F is a homeomorphism onto a neighborhood of γ.

The above steps were carried out in [1] for the H^s_δ case, and the generalization to the present asymptotic conditions is straightforward.

Remark. The construction of an ILH structure for the quotient spaces $\mathbf{M}^\infty/D^\infty$ and $\mathbf{M}^\infty/D^\infty$ has not been worked out in detail for the case of noncompact M, but should be straightforward following the work of Bourgignon [12]. The following is the final result concerning the structure of the quotient spaces:

(1) The space $\mathbf{M}_{\mathrm{D}} = \mathbf{M}^\infty/\mathbf{D}^\infty$ is a strong ILH variety with singularities corresponding to the metrics in $\mathbf{M}^\infty$ with isometries.

(2) The space $\mathbf{M}_D = \mathbf{M}^\infty/D^\infty$ is a strong ILH manifold (this reflects the fact that D does not contain any isometries for metrics in $\mathbf{M}$). There is a smooth action of the Euclidean group E on $\mathbf{M}_D$ and the quotient $\mathbf{M}_D/E = \mathbf{M}_{\mathrm{D}}$. ∎

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